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Steady flow through an asymmetric wavy channel: Non-orthogonal coordinates

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Abstract: In this paper we consider the steady flow of a viscous fluid through a channel bounded by two sinusoidally varying plates in same phase and separated by a mean distance *2h*. For the non-varying channel, the classical parabolic velocity profile for the fully developed flow is well known. An attempt here is made to analyze the flow in a generalized non-orthogonal coordinate system that renders the wavy channels as plane walls. Continuity equation and Navier-Stokes equations are presented in the generalized coordinate system and simplified through use of small perturbation under small Reynolds number approximation. Centerline velocity have been evaluated and discussed.

Key words: Wavy Channel, Navier-Stoke equations, Small Reynolds Number, Perturbation, Drag

Introduction

Walls making up fractured rocks are not plane but may be taken to be wavy, thus wavy channel flow may be of significance in geophysical fluid flow problems. It has lots of applications in other fields too *e.g.* in oil recovery, biological transport processes, polymeric composite manufacturing and for enhancement of heat transfer in heat exchangers. Flow in sinusoidal wavy wall has been analyzed amongst others by Benjamin (1959), Kim (2001), Lyne (1971), Montalbano *et al*. (1998), Patel *et al.* (1991), Ralph (1987) and Wang (2004). Sankar and Sinha (1976) studied the Rayleigh problem for a wavy wall by applying conformal mapping technique to transform the wavy boundary to a regular one. Thomas et al. [9], neglecting inertia terms and using perturbation method that transforms a problem with irregular boundaries into a series of problems with plane boundaries, have investigated the flow profile and power requirements in an oscillatory flow field in a wavy walled channel.

But this paper is different from others as here generalized non-orthogonal coordinate system is used rendering directly the wavy channel walls to plane ones. The method may be extended from sinusoidally varying walls to walls with arbitrary fluctuation.

In generalized coordinate system with coordinates x^i (i = 1,2), the equation of continuity for incompressible steady flow is given by

$$\frac{\partial v^i}{\partial x^i} + v^k \left\{ \begin{matrix} i \\ k\ i \end{matrix} \right\} = 0 \qquad \text{................. (1.1)}$$

and Navier-Stokes equations with no external force are given by Sokolnikoff (1964).

$$v^j \left(\frac{\partial v^i}{\partial x^j} + \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\} v^l \right) = g^{jk} \left(\frac{\mu}{\rho} \right) \left[\frac{\partial^2 v^i}{\partial x^j \partial x^k} + \left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} \frac{\partial v^l}{\partial x^j} + \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\} \frac{\partial v^l}{\partial x^k} - \left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} \frac{\partial v^i}{\partial x^l} \right]$$
$$+ g^{jk} \left(\frac{\mu}{\rho} \right) \left[\frac{\partial}{\partial x^k} \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\} + \left\{ \begin{matrix} i \\ m\ k \end{matrix} \right\} \left\{ \begin{matrix} m \\ l\ j \end{matrix} \right\} - \left\{ \begin{matrix} i \\ l\ m \end{matrix} \right\} \left\{ \begin{matrix} m \\ j\ k \end{matrix} \right\} \right] v^l - \frac{g^{ik}}{\rho} \frac{\partial p}{\partial x^k} \qquad \text{......... (1.2)}$$

where v^i are contravariant components of velocity, p is the pressure, μ is coefficient of viscosity, ρ is the density of fluid at any point and also curly bracket { } represents Christoffel symbol of second kind.

Equations governing incompressible wavy channel flow in generalized coordinates: Now, consider the flow through a channel bounded by two wavy surfaces.

$$Y = h\,(\pm 1 + \delta Sin\,\lambda X) \qquad \text{(2.1)}$$

where 2h is the mean distance between the two surfaces, λ is 2π times the wave number and (X, Y) are Cartesian coordinates with X - axis along the central line of the channel.

We introduce the transformation

$$\left.\begin{aligned} X &= hx^1 \\ Y &= h\left[x^2 + \delta Sin\left(\lambda h x^1\right)\right] \end{aligned}\right\}$$

so that the channel boundaries assume the simple form $x^2 = \pm 1$.

Observe that x^i (i = 1,2) are non-orthogonal curvilinear coordinates. For convenience, we denote $Sin\,(\lambda h x^1) = S$,

$Cos\left(\lambda h x^1\right) = C$ and $\lambda h = \Lambda$

The metric tensor g^{ij} and g_{ij} are then obtained as

$$g_{ij} = \begin{pmatrix} h^2 & \delta h^2 \Lambda C & 0 \\ \delta h^2 \Lambda C & h^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix}$$

and

$$g^{ij} = \begin{pmatrix} h^{-2} & -\delta h^{-2} \Lambda C & 0 \\ -\delta h^{-2} \Lambda C & h^{-2} & 0 \\ 0 & 0 & h^{-2} \end{pmatrix} \qquad \text{(2.3)}$$

$$g = \left|g_{ij}\right| = h^4$$

Using tensor methods, assuming δ to be a small perturbation and neglecting O (δ^2) terms, equation of continuity and Navier–Stokes equations for steady flow become

$$\left(\frac{\partial v^1}{\partial x^1}\right) + \left(\frac{\partial v^2}{\partial x^2}\right) = 0 \qquad \text{(2.4)}$$

$$\left(\frac{\partial v^1}{\partial x^1}v^1+\frac{\partial v^1}{\partial x^2}v^2\right)$$

$$=\left(\frac{\nu}{h^2}\right)\left(\frac{\partial^2 v^1}{\left(\partial x^1\right)^2}+\frac{\partial^2 v^1}{\left(\partial x^2\right)^2}-2(\delta\Lambda C)\frac{\partial^2 v^1}{\partial x^1\partial x^2}+\delta\Lambda^2\mathrm{S}\frac{\partial v^1}{\partial x^2}\right)+\frac{1}{\rho h^2}\left(\delta\Lambda\mathrm{C}\frac{\partial p}{\partial x^2}-\frac{\partial p}{\partial x^1}\right) \quad \text{............ (2.5)}$$

$$\left(\frac{\partial v^2}{\partial x^1}v^1+\frac{\partial v^2}{\partial x^2}v^2-\delta\Lambda^2 S\left(v^1\right)^2\right)-\left(\frac{\delta\Lambda C}{\rho h^2}\frac{\partial p}{\partial x^1}-\frac{1}{\rho h^2}\frac{\partial p}{\partial x^2}\right)$$

$$=\frac{\nu}{h^2}\left[\frac{\partial^2 v^2}{\left(\partial x^1\right)^2}+\frac{\partial^2 v^2}{\left(\partial x^2\right)^2}-2\delta\Lambda C\frac{\partial^2 v^2}{\partial x^1\partial x^2}+\delta\left\{\left(\Lambda^2\mathrm{S}\right)\frac{\partial v^2}{\partial x^2}-2\left(\Lambda^2\mathrm{S}\right)\frac{\partial v^1}{\partial x^1}-v^1\left(\Lambda^3 C\right)\right\}\right] \quad \text{..... (2.6)}$$

These equations have to be solved under the no slip condition and vanishing of the normal velocity at $x^2 = \pm 1$

Perturbation solution: Assuming δ to be small and introducing dimensionless new variables u^o, u^1, w^1, p^o and p^1 as follows:

$$\left.\begin{aligned} v^1 &= \left(\frac{U}{h}\right)\left(u^0+\delta u^1\right)\\ v^2 &= \left(\frac{U}{h}\right)\delta w^1\\ p &= \left(p^0+\delta p^1\right)\rho U^2 \end{aligned}\right\} \quad \text{...................................... (3.1)}$$

Observe that the contravariant velocity components v^i and v^2 have same dimensions as $\frac{U}{h}$ and pressure p has same dimension as. pU^2Here U represents the centerline velocity corresponding to straight channel which is linearly related to the corresponding pressure gradient. Substituting (3.1) in equation (2.4), (2.5) and (2.6) we get two sets of three equations as follows:

Zero order equation:

$$\left.\begin{aligned} &\left(\frac{\partial u^0}{\partial x^1}\right)=0\\ &\left(\frac{\partial p^0}{\partial x^2}\right)=0\\ &\left\{\frac{\partial^2 u^0}{\left(\partial x^2\right)^2}\right\}=R\left(\frac{\partial p^0}{\partial x^1}\right) \end{aligned}\right\} \quad \text{................................ .... (3.2)}$$

Where $R = \dfrac{\rho h U}{\mu}$ is Reynolds number.

The zero order solution with the boundary conditions $u^o(\pm 1) = 0$ provides

$$u^0 = \{1-(x^2)^2\} \qquad \text{.......... (3.3)}$$

$$\left(\frac{\partial p^0}{\partial x^2}\right) = 0 \qquad \text{.......... (3.4)}$$

$$\left(\frac{\partial p^0}{\partial x^1}\right) = -\frac{2}{R} \qquad \text{.......... (3.5)}$$

First order perturbation equations: Using solution [(3.3) - (3.5)], the first order equations are expressible as:

$$\left(\frac{\partial u^1}{\partial x^1}\right) + \left(\frac{\partial w^1}{\partial x^2}\right) = 0 \qquad \text{.......... (3.6)}$$

$$\frac{\partial p^1}{\partial x^1} = \frac{1}{R}\frac{\partial^2 u^1}{\left(\partial x^1\right)^2} + \frac{1}{R}\frac{\partial^2 u^1}{\left(\partial x^2\right)^2} - \left(u^0\right)\left(\frac{\partial u^1}{\partial x^1}\right) + 2x^2 w^1 - 2x^2\Lambda^2 \mathrm{S}\frac{1}{R} \qquad \text{.......... (3.7)}$$

$$\left(\frac{\partial p^1}{\partial x^2}\right) = \frac{1}{R}\left[\frac{\partial^2 w^1}{\left(\partial x^1\right)^2} + \frac{\partial^2 w^1}{\left(\partial x^2\right)^2} - u^0\Lambda^3 C\right] - \left(\frac{\partial w^1}{\partial x^1}\left(u^0\right) - \Lambda^2 S\left(u^0\right)^2 + \frac{2\Lambda C}{R}\right) \qquad \text{.......... (3.8)}$$

These equations have been solved under following boundary conditions: $v^1 = 0$ and $v^2 = 0$ at $x^2 = \pm 1$

For small values of Reynolds numbers R, the solution is

$$v^1 = \left(\frac{U}{h}\right)\left[\left\{1-\left(x^2\right)^2\right\} + \delta\left[\frac{2}{\left(\Lambda + h_c h_s\right)}\left\{h_c\Lambda\left(x^2\right) Cosh\left(\Lambda x^2\right) + \left(h_c - h_s\Lambda\right) Sinh\left(\Lambda x^2\right)\right\} - 2\left(x^2\right)\right] Sin\left(\Lambda x^1\right)\right]$$

$$\text{.......... (3.9)}$$

$$v^2 = \delta\Lambda\left(\frac{U}{h}\right)\left[\frac{2}{\left(\Lambda + h_c h_s\right)}\left\{h_s Cosh\left(\Lambda x^2\right) - h_c\left(x^2\right)Sinh\left(\Lambda x^2\right)\right\} - u^0\right]Cos\left(\Lambda x^1\right) \quad \text{......... (3.10)}$$

where : $h_c = Cosh(\Lambda)$ and $h_s = Sinh(\Lambda)$

For obtaining the physical results the solution has been transformed to the non-orthogonal physical components, which up to first order perturbation in δ are as follows :

$$V(1) = U\left[\left\{1-\left(x^2\right)^2\right\} + \delta\left[\frac{2}{\left(\Lambda + h_c h_s\right)}\left\{h_c\Lambda\left(x^2\right)Cosh\left(\Lambda x^2\right) + \left(h_c - h_s\Lambda\right)Sinh\left(\Lambda x^2\right)\right\} - 2\left(x^2\right)\right]Sin\left(\Lambda x^1\right)\right] \quad \text{.... (3.11)}$$

$$V(2) = \delta U\Lambda\left[\frac{2}{\left(\Lambda + h_c h_s\right)}\left\{h_s Cosh\left(\Lambda x^2\right) - h_c\left(x^2\right)Sinh\left(\Lambda x^2\right)\right\} - u^0\right]Cos\left(\Lambda x^1\right) \quad \text{............ (3.12)}$$

and corresponding orthogonal Cartesian components are:

$$V_c(1) = U\left[\left\{1-\left(\frac{Y}{h}\right)^2\right\} + \delta\left\{\frac{2h_c\Lambda Y}{h\left(\Lambda + h_c h_s\right)}Cosh\left(\frac{\Lambda Y}{h}\right) + \frac{2\left(h_c - h_s\Lambda\right)}{\left(\Lambda + h_c h_s\right)}Sinh\left(\frac{\Lambda Y}{h}\right)\right\}Sin\left(\frac{\Lambda X}{h}\right)\right] \quad \text{........ (3.13)}$$

$$V_c(2) = \frac{2\delta U\Lambda}{\left(\Lambda + h_c h_s\right)}\left\{h_s Cosh\left(\frac{\Lambda Y}{h}\right) - h_c\left(\frac{Y}{h}\right)Sinh\left(\frac{\Lambda Y}{h}\right)\right\}Cos\left(\frac{\Lambda X}{h}\right) \quad \text{.................. (3.14)}$$

Flow characteristics -

Center line velocity:

Letting $Y = \delta hS$ in (3.13) and (3.14), neglecting O(δ^2) terms, we get axial and transverse components of central line velocity as

$$\left.\begin{aligned} &V_X(c) = U \\ \text{and} \quad &V_Y(c) = \frac{2\delta h_s U\Lambda}{\left(\Lambda + h_c h_s\right)}Cos\left(\frac{\Lambda X}{h}\right)\end{aligned}\right\} \quad \text{.. (4.1)}$$

respectively. It is observed that axial component has the same constant value as in the corresponding straight channel case.

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